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Stability of spiralling solitary waves in Hamiltonian systems

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We present a rigorous criterion for stability of spiralling solitary structures in Hamiltonian systems incorporating the angular momentum integral and demonstrate its applicability to the spiralling of two mutually incoherent optical beams propagating in a photorefractive material.

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The richness of spatiotemporal dynamics of light in nonlinear media has attracted much attention over the last decade. One of the many recent areas of interest is spiralling of self-localized beams of light [1]. Existence of spiralling structures in conservative nonlinear models was reported over a decade ago in the context of generalized Klein-Gordon [2] and nonlinear Schrödinger [3,4] equations. Recently there has been renewed interest, following a series of experiments demonstrating spiralling of a pair of mutually incoherent light beams propagating in photorefractive materials [5–8], and in concomitant numerical modeling [6–9] broadly supporting these experiments. The basic idea is that the attractive forces between two optical beams can be compensated by a centrifugal force due to nonzero angular momentum of the beams [4,5]. Later developments have also shown seemingly stable spiralling structures resulting from azimuthal instability of a vortex beam within the potential created by a strong soliton field incoherent with the vortex [8,9]. These structures have been termed *propeller solitons* [8] or *rotating dipole-mode vector solitons* [9].

On the analytical side two main approaches have been pursued to describe spiralling. One is a variational approach [10,11], which is known to produce good approximations for the shape of solitary solutions. Variational methods, however, cannot reliably predict the spectral properties of solitons [12]. Substituting approximate solutions into the Hamiltonian and deriving an expression for the effective interaction potential [13,14] can correctly predict stability with respect to changes of the soliton separations, but cannot examine stability with respect to the phase degrees of freedom, because a self-consistent potential function cannot be defined if the latter are fully accounted for.

A more rigorous approach is based on asymptotic theory of weakly interacting solitary waves [2,7]. It yields all relevant eigenvalues and therefore can be expected to give good predictions for stability. However, both linearization near the spiralling solution of the reduced ordinary differential equations for the soliton parameters and finding this solution in analytical form are often very difficult problems on their own. Comparison of numerical modeling of the full partial and reduced ordinary differential equations is often the only sensible option [7]. Obviously neither of the above methods is applicable to describe stability of spiralling structures which cannot be approximated by a set of weakly overlapping individual solitons.

Thus obtaining a rigorous and general analytical criterion for the stability of spiralling solutions in Hamiltonian models evidently is an open and important problem, which we approach and solve in this work. Results presented below are rather general, but we derive them in the context of two coupled nonlinear Schrödinger (NLS) equations with saturable nonlinearity, which have well established spiralling solutions [7–9], and are believed to provide a reasonable approximation to the nonlinear interaction of incoherent beams in photorefractives [5,7,8]

$$(i\partial_z+\partial_x^2+\partial_y^2)E_{1,2}-f(I)E_{1,2}=0, \tag{1}$$

where $f(I)=1/(1+I)$, $I=|E_1|^2+|E_2|^2$, and z and x,y are, respectively, dimensionless longitudinal and transverse coordinates. We consider solutions which spiral with constant angular frequency ω in propagation along z. As in recent publications on Bose-Einstein condensation in rotating traps (see, e.g., Ref. [15]), we change to a rotating coordinate system: $E_{1,2}(x,y,z)=F_{1,2}(X,Y,z)e^{i\kappa_{1,2}z}$ with $X=x\cos(\omega z)+y\sin(\omega z)$, $Y=-x\sin(\omega z)+y\cos(\omega z)$. The wave-number corrections $\kappa_{1,2}$, together with ω, parametrize the solutions of interest. Now Eqs. (1) read as

$$[i\partial_z+\omega\hat{L}+\partial_X^2+\partial_Y^2]F_{1,2}-[\kappa_{1,2}+f(I)]F_{1,2}=0, \tag{2}$$

where $\hat{L}=-i(X\partial_Y-Y\partial_X)=i\partial_\theta$ is the z component of the angular momentum operator, familiar in quantum mechanics, and $\theta=\arg(X+iY)$. The convenience of the form (2) is that beams spiralling with frequency ω in the (x,y) frame are stationary in the (X,Y) frame and that parameterization by ω and $\kappa_{1,2}$ is now explicit.

By analogy with stability thresholds known for other multiparameter solitary waves [16], which in their turn stem from the seminal Vakhitov-Kolokolov criterion [17], we can already guess the expression for the stability threshold for such a solution ($F_{1,2}$ localized in X,Y). This condition is

$$D_0=\det\begin{pmatrix}\frac{\partial P_1}{\partial\kappa_1} & \frac{\partial P_1}{\partial\kappa_2} & \frac{\partial P_1}{\partial\omega}\\ \frac{\partial P_2}{\partial\kappa_1} & \frac{\partial P_2}{\partial\kappa_2} & \frac{\partial P_2}{\partial\omega}\\ \frac{\partial L}{\partial\kappa_1} & \frac{\partial L}{\partial\kappa_2} & \frac{\partial L}{\partial\omega}\end{pmatrix}=0, \tag{3}$$

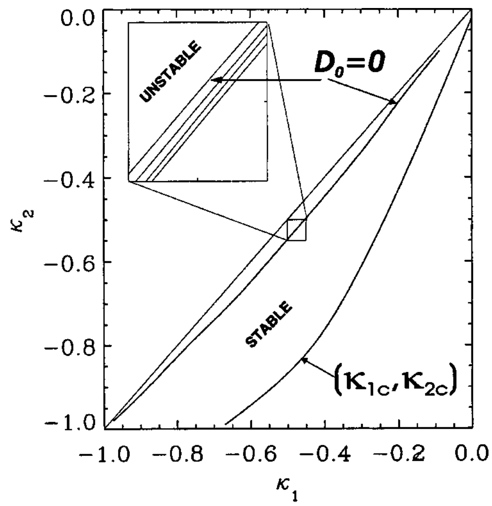

FIG. 1. Diagram showing region of existence and stability of the spiralling solutions in the (κ_1,κ_2) plane, for different values of ω. Interchanging E_1 and E_2, one can plot a symmetric diagram in the region $\kappa_2>\kappa_1$. Inset shows $D_0=0$ lines corresponding to $\omega=0$, 0.02, 0.03, and 0.04, which are plotted from left to right, respectively. $D_0=0$ lines marked by arrows correspond to $\omega=0$.

where $P_{1,2}=\int dXdY|F_{1,2}|^2$ are the independently conserved power flows of the two interacting fields and $L=\vec{i}_z\cdot\int\int dXdY\vec{r}\times\vec{j}=\Sigma_{n=1,2}\int\int rdrd\theta\ \mathrm{Im}(F_n^*\partial_\theta F_n)$ is the orbital angular momentum integral. Here $\vec{r}=\vec{i}_XX+\vec{i}_YY$, $\vec{j}=\Sigma_{n=1,2}[F_n^*(\vec{i}_X\partial_X+\vec{i}_Y\partial_Y)F_n-\mathrm{c.c.}]/(2i)$. $\vec{i}_X,\vec{i}_Y,\vec{i}_z$ are the unit vectors along X, Y, and z axes. The conservation laws $\partial_zL=\partial_zP_{1,2}=0$ follow directly from the Hamiltonian nature of our model combined with invariance of Eqs. (2) with respect to rotations in the (X,Y) plane and to the two phase shifts $(F_1,F_2)\to(F_1e^{i\phi_1},F_2e^{i\phi_2})$.

To formally derive Eq. (3) and verify its applicability to the stability of spiralling solutions of Eqs. (1), we adopt the following approach: first, we prove, both analytically and numerically, that stationary, $\omega=0$, dipole-mode soliton solutions of Eqs. (2) can be smoothly continued along ω; second, we derive an expression for the eigenvalues governing stability of this solution and numerically verify change of stability at $D_0=0$; third, we discuss applicability and generalization of our results for other cases and models.

It has been previously established that stationary dipole-mode solitons and rotating dipoles [8,9] bifurcate from the scalar fundamental soliton solution, $F_1=A_0(r=\sqrt{X^2+Y^2})$, $F_2=0$, for certain values of $(\kappa_1,\ \kappa_2)$. Linearization of Eqs. (2) near this soliton leads to a factorizable eigenvalue problem. The modes of the excitations of the F_2 component are related to the eigenstates of the operator $\partial_X^2+\partial_Y^2-\kappa_2-1/(1+A_0^2)$, which has eigenfunctions of the form $f_m(r)e^{\pm im\theta}$, $m=0,1,2,\ldots$. Scanning the (κ_1,κ_2) plane, one can show that the scalar soliton always remains stable, but that there are special lines, where eigenvalues corresponding to F_2 eigenstates with a particular value of m cross zero. That corresponding to $m=1$ marks the boundary line, $(\kappa_{1c},\kappa_{2c})$, where dipole-mode solitons bifurcate from the scalar soliton, see Fig. 1. Thus the weak F_2 component of the emerging two-component solutions can be seen as a field guided by the strong F_1 component. To obtain expressions for the bifurcating solutions we use asymptotic expansions $F_1=A_0(r)+O(\epsilon^2)$, $F_2=\epsilon B_1(z,r,\theta)+O(\epsilon^3)$, where $\epsilon\ll 1$ measures distance from the bifurcation line. Assuming $\omega\sim\epsilon^2$ one can show that $\epsilon B_1=f(r)[a_+(z)e^{i\theta}+a_-(z)e^{-i\theta}]$, where $f(r)$ vanishes at zero and infinity and $a_\pm$ obey

$$i\partial_z a_\pm=a_\pm(\alpha|a_\pm|^2+\beta|a_\mp|^2)\pm\omega a_\pm. \tag{4}$$

The self- and cross-phase modulation constants α,β can be found only numerically. The term in ω originates from $\hat{L}$ in Eqs. (2). This system represents the normal form describing breaking of rotational symmetry in Hamiltonian models and is generic in the vicinity of a point where a rotationally non-invariant solution bifurcates from a rotationally invariant one [18].

The solution of interest to Eqs. (4) is

$$a_\pm=\tilde{a}_\pm e^{i\tilde{\kappa}_2 z},\quad |\tilde{a}_\pm|^2=\frac{\tilde{\kappa}_2(\beta-\alpha)\mp\omega(\alpha+\beta)}{\alpha^2-\beta^2}, \tag{5}$$

where $\tilde{\kappa}_2=(\kappa_2-\kappa_{2c})\sim\epsilon^2$. This solution exists for $\tilde{\kappa}_2>0$ and in the ω interval

$$\frac{\tilde{\kappa}_2(\beta-\alpha)}{\beta+\alpha}<\omega<\frac{\tilde{\kappa}_2(\alpha-\beta)}{\beta+\alpha}, \tag{6}$$

where $0<(\alpha-\beta)/(\alpha+\beta)<1$. In the original frame, for $\omega=0$ it describes stationary, and, for $\omega\neq 0$, rotating or spiralling dipoles. At the boundaries of its existence (6) this solution is a vortex in F_2, with either a_+ or a_- equal to zero. For more details on localized vortex solutions in multicomponent NLS models see, e.g., Refs. [9,19].

Thus we have demonstrated analytically that a spiralling solution is parametrized by its frequency ω and, therefore, derivatives $\partial_\omega P_{1,2}$, $\partial_\omega L$ indeed exist. Note that equations analogous to Eqs. (4) have been also derived in Ref. [8], but this link between the stationary and rotating dipoles following from them was not discussed. Several other new and important consequences of these equations are presented below.

To find spiralling solutions arbitrarily far from the bifurcation boundary we have solved Eqs. (2) with $\partial_z=0$ using a numerical technique based on a Newton method. We obtained such solutions throughout the region bounded by the lines $\kappa_{1,2}=\kappa_{1c,2c}$ and $\kappa_1=\kappa_2$ in the (κ_1,κ_2) plane, see Fig. 1. By symmetry, corresponding solutions with E_1 and E_2 interchanged exist in the mirror-image domain $\kappa_2>\kappa_1$.

For $\kappa_{1,2}$ values far enough from $\kappa_{1c,2c}$, the F_1 field develops two strong intensity peaks overlapping with those of the F_2 dipole. In this region a spiralling structure can be interpreted as a dynamical bound state of two weakly overlapping single-hump vector solitons [7]. Thus our results indicate that the spiralling solitons found in Ref. [7] and the rotating dipole (or propeller) solitons [8,9] belong to the same soliton family.

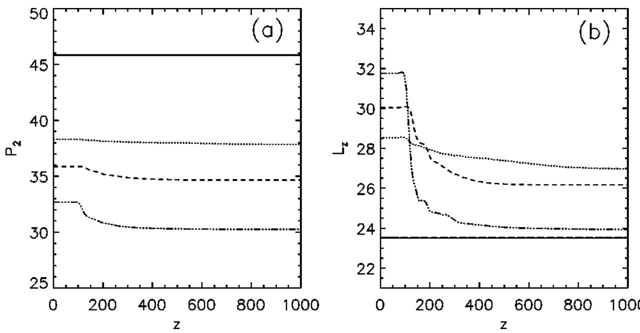

FIG. 2. z evolution of P_2 and L. $\kappa_2 = -0.7$. Solid, dotted, dashed, and dash-dotted lines, respectively, denote, $\kappa_1 = -0.63$, -0.670, -0.675, and -0.680.

Having established existence of stationary solutions of Eqs. (2) parametrized by ω and $\kappa_{1,2}$ we now consider their stability. Equations (4) predict that both spiralling dipoles and vortex solutions are stable. However, these equations do not capture possible instabilities due to angular harmonics with $|m| \neq 1$ and are only valid close to $(\kappa_{1c}, \kappa_{2c})$. The straightforward method to study stability of the spiralling solution $F_{1,2} = F_{1s,2s}$ found from Eqs. (2) is to set $F_{1,2} = F_{1s,2s}(X,Y) + [u_{1,2}(X,Y)e^{\lambda z} + i w_{1,2}(X,Y)e^{\lambda z} + \text{c.c.}]$ and linearize Eqs. (2) assuming that $u_{1,2}, w_{1,2}$ are small. The next step is to find the spectrum of the resulting eigenvalue problem $\hat{\mathcal{J}}\vec{u} = \lambda \vec{u}$, where $\vec{u} = (u_1, u_2, w_1, w_2)^T$. The explicit form of of the operator $\hat{\mathcal{J}}$ is too cumbersome to be presented here, and reliable numerical analysis of its spectrum is a formidable computational problem in its own right. Therefore, we will rely in what follows on a combination of analytical techniques and direct numerical modeling of Eqs. (1).

We have undertaken extensive numerical modeling of Eqs. (1), initialized with our computed spiralling solutions with small added noise. For various values of $\kappa_{1,2}$ and ω, we observe unstable behavior in the vicinity of the line $\kappa_1 = \kappa_2$. The unstable dynamics was monitored by plotting the z evolution of the powers $P_{1,2}$ and angular momentum L, see Fig. 2. Any instability resulting in the radiation of energy leads to the decay of these quantities, because we used absorbing boundary conditions on the perimeter of the computational window, where the solitonic field is negligible. Figure 2 shows several such plots for $\kappa_2 = -0.7$ and several values of κ_1. $P_{1,2}$, L are conserved for $\kappa_1 > \sim -0.65$, but closer to the line $\kappa_1 = \kappa_2$, their evolution indicates shedding of radiation with subsequent stabilization at new stationary levels. Corresponding volume plots are shown in Fig. 3.

We interpret this as an intrinsic instability of the spiralling solutions. Because $\kappa_{1,2}$ and ω parameterize a particular solution, not the system as a whole, they can change during the evolution of an unstable solution. Therefore, as Figs. 2 and 3 illustrate, unstable spiralling solutions with given values of $P_{1,2}$ and L can evolve into stable spiralling solutions carrying different powers and angular momentum, plus some nonsolitonic radiation.

By numerically computing the properties of whole families of spiralling solutions we have been able to evaluate the determinant D_0 given by Eq. (3). We find that it does indeed

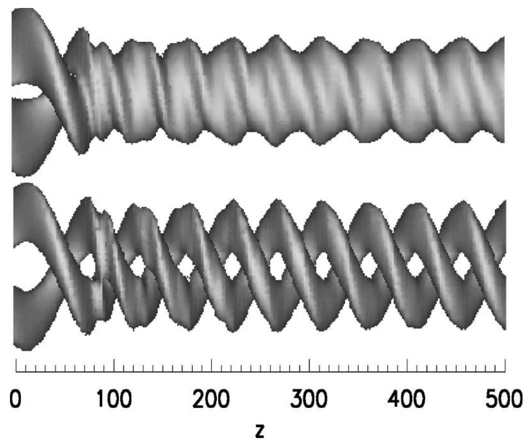

FIG. 3. Volume plots showing instability induced dynamics of $|E_1|$ (top) and $|E_2|$ (bottom) for initial condition corresponding to $\kappa_1 = -0.68$, $\kappa_2 = -0.7$, $\omega = 0.04$. One can observe an instability induced increase of the frequency to $\omega \simeq 0.065$. Surfaces plotted correspond to $|E_{1,2}| = 0.5$.

change sign at the onset of instability. D_0 is positive in the region where numerical modeling of Eqs. (1) indicates stable spiralling, and negative in the region where unstable dynamics is observed. We now prove that change of sign of D_0 is a sufficient condition for the existence of instability. To derive this criterion formally we solve the eigenvalue problem $\hat{\mathcal{J}}\vec{u} = \lambda \vec{u}$, assuming that $|\lambda|$ is a small parameter, and taking $\vec{u}$ as a superposition of the neutral (or Goldstone) eigenmodes of $\hat{\mathcal{J}}$ [16]. These neutral modes can be found by applying infinitesimal symmetry transformations to the soliton solution. In our case the two phase symmetries generate the neutral modes $\vec{u}_{\phi_1} = (-\text{Im } F_{1s}, 0, \text{Re } F_{1s}, 0)^T$, $\vec{u}_{\phi_2} = (0, -\text{Im } F_{2s}, 0, \text{Re } F_{2s}, 0)^T$, while the rotation symmetry in the (X,Y) plane generates $\vec{u}_\theta = \partial_\theta(\text{Re } F_{1s}, \text{Re } F_{2s}, \text{Im } F_{1s}, \text{Im } F_{2s})^T$. Thus we set $\vec{u} = \vec{u}_{cr} = C_1 \vec{u}_{\phi_1} + C_2 \vec{u}_{\phi_2} + C_3 \vec{u}_\theta + O(|\lambda|)$, where $C_{1,2,3} = O(1)$ are some constants which can be found at higher order. Developing perturbation theory up to the fourth order in $|\lambda|$, (analogous to the theory described in Ref. [16]) we have found that eigenvalues corresponding to $\vec{u}_{cr}$ are given by $\lambda_{cr}^2 = -D_0/D_1 + O(|\lambda_{cr}|^4)$. The expression for D_1 is quite cumbersome, but can be deduced from the corresponding two-parameter formulas given in Ref. [16]. Provided D_1 is finite, our expression for λ_{cr}^2 formally demonstrates that spiralling solutions always have at least one unstable mode in any neighborhood of $D_0 = 0$. Direct modeling of Eqs. (1) indicates that $D_0 = 0$ marks the *first* instability threshold met as spiralling solutions are tracked in parameter space (see Fig. 1) from the bifurcation line $(\kappa_{1c}, \kappa_{2c})$, where they emerge towards the line $\kappa_1 = \kappa_2$.

Our expression (3) for D_0 is in fact a new generalized form of the Vakhitov-Kolokolov criterion [16,17], incorporating the angular momentum integral. This is a direct consequence of the need to include the rotational mode $\vec{u}_\theta$ in the derivation of D_0. The relevance of this neutral mode in our case can be seen by noting that Eqs. (4) are invariant with respect to the two phase shifts $(a_+,a_-) \to (a_+ e^{i\phi_2+i\psi}, a_- e^{i\phi_2-i\psi})$. The phase ϕ_2 is the same as previously introduced for Eqs. (2), but the physical meaning of ψ is an angle of rotation in the (X,Y) plane. The solution $\tilde{a}_\pm=0$, corresponding to the scalar soliton, is itself rotationally invariant. For the pure vortex solitons any change of ψ can be mimicked by a shift of ϕ_2. Thus for these two classes of solutions rotational symmetry is not broken and so the corresponding neutral mode is absent from their spectra. Rigorous theory explaining absence of neutral modes in the situations analogous to those considered here was recently published in Ref. [20]. This absence can be easily verified by direct linearization of Eqs. (4). The dipole solution with both $\tilde{a}_\pm \neq 0$, however, breaks rotational symmetry and therefore acquires an extra neutral eigenmode, which in turn implies an additional dimension in the determinant D_0. For a general discussion of symmetry breaking in the context of soliton theory, see, e.g., Ref. [21].

Note that $\kappa_1=\kappa_2$ is a critical line, where the equations for $F_{1,2}$ become identical and neither of the components can be identified as guiding or guided [22]. Though detailed studies of the vicinity of the line $\kappa_1=\kappa_2$ are left for the future, we expect that solutions discussed here are in some way linked with another class of spiralling structures recently found in the single NLS equation with saturable nonlinearity [13], which reduces to Eqs. (1) for $\kappa_1=\kappa_2$. Though sufficiency of the condition $D_0/D_1<0$ for these solutions to be unstable is clear, its necessity is a much more subtle question, which requires separate detailed investigation.

In conclusion, we have presented and proved a generalized form of the Vakhitov-Kolokolov criterion which incorporates the angular momentum integral and is generally applicable in Hamiltonian systems exhibiting breaking of the rotational symmetry. We have demonstrated that this criterion correctly predicts instability thresholds for spiralling solutions of the saturable vector Kerr model, which describes recent experimental observations in photorefractive media [23].

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